Hyperelliptic Solutions of KdV and KP equations: Reevaluation of Baker's Study on Hyperelliptic Sigma Functions

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Abstract

Explicit function forms of hyperelliptic solutions of Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) equations were constructed for a given curve $y^2 = f(x)$ whose genus is three. This study was based upon the fact that about one hundred years ago (Acta Math. (1903) **27**, 135-156), H. F. Baker essentially derived KdV hierarchy and KP equation by using bilinear differential operator $\mathbf{D}$, identities of Pfaffians, symmetric functions, hyperelliptic σ-function and $\wp$-functions; $\wp_{\mu\nu} = -\partial_\mu\partial_\nu \log\sigma = -(\mathbf{D}_\mu\mathbf{D}_\nu\sigma\sigma)/2\sigma^2$. The connection between his theory and the modern soliton theory was also discussed.

§1. Introduction

In this article we will construct explicit function forms of hyperelliptic solutions of Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) equations for a given curve $y^2 = f(x)$ whose genus is three, along the lines of the study of H. F. Baker's sigma function [B1, B2, B3]. This construction means reevaluation of Baker's studies on hyperelliptic functions which were done one hundred years ago as a special case of his studies of algebraic functions over a general compact Riemannian surface [B3]. Although his general theory has been already known as the studies related to Baker-Akhiezer functions [B1, K1, K2], the paper [B3] published in 1903 might have been left behind.

According to [B3], around 1898 he discovered series of partial differential equations which lead hyperelliptic sigma function, σ, and $\wp$-functions, $\wp_{\mu\nu} := \partial_\mu\partial_\nu \log\sigma$. If one saw the partial differential equations, he would know that they are related to soliton equations such as the KdV equations or the KP equations. However Baker's definition of parameters is twisted from those in modern soliton theory. Further as the paper [B3] requires knowledge of hyperelliptic σ and $\wp$ functions which might not be familiar nowadays [B1, B2, Ô2], it is not easy to understand its contents and to confirm the derivation. In this paper, we will give correspondences between his differential equations and, the KP equation and first and second equations of the KdV hierarchy in order to construct explicit function forms of their periodic multi-soliton solutions.

The identification between Baker's differential equations and these soliton equations means that Baker essentially discovered the KdV hierarchy and the KP equation one

hundred years ago. In the study, he used the Pfaffian, symmetric functions, bilinear operator $\mathbf{D}$, hyperelliptic sigma function σ and $\wp$-functions; $\wp_{\mu,\nu} = -(\mathbf{D}_\mu \mathbf{D}_\nu \sigma\sigma)/2\sigma^2$.

In this paper, we will comment on its relation to soliton theory in Sec 4. As we mentioned there, we can regard that Baker's theory is on the differentials of the first kind over a hyperelliptic curve. As compared with his theory, the ordinary soliton theories *e.g.*, Sato theory [SS], Date-Jimbo-Kashiwara-Miwa (DKJM) theory [DKJM], Krichever theory [K1, K2], conformal field theory [KNTY] and so on, can be considered as theories of the differentials of the second kind. Thus Baker's theory is not directly connected with the modern soliton theories, even though he used the Pfaffian, symmetric functions, bilinear operator $\mathbf{D}$. Indeed he might be interested only in properties of periodic functions on non-degenerate curves. As long as I know, he did not consider the soliton solutions, which is expressed by hyperbolic functions or trigonometric functions. Hence he neither reached Hirota's direct method [H] even though he defined and used the bilinear operator.

However, as all values appearing in Baker's theory have algorithms to evaluate themselves, we can deal with hyperelliptic functions in the framework of his theory as we can do with elliptic functions. For example, we can concretely determine any coefficients of Laurent or Taylor expansions of σ and $\wp$ functions at any points in any hyperelliptic curves [B1, B2, B3, G, Ô1, Ô3]. Recently requests to evaluate the hyperelliptic functions explicitly appear from various fields, *e.g.*, from study on the Abel functions, from number theory [G, Ô1, Ô3], and from study of an elastica which is closely related to the KdV equations [Ma1, Ma2]. There Baker's theory of hyperelliptic functions plays a central role [G, Ô1, Ô3, Ma2]. The purpose of this article is to reevaluate Baker's work from the viewpoint of soliton theory.

After completion of this article, I knew the works of Buchstaber, Enolskii and Leykin [BEL1-3] and others [CEEK, EE, EEL, EEP, N and references therein]. The authors in [BEL1-3,CEEK, EE, EEL, EEP, N] also reevaluated theory of Baker's hyperelliptic sigma functions, which they call Kleinian functions, and have extended it from point of view of soliton theory. For example in [B3], Baker derived a differential identity of the hyperelliptic $\wp$-functions of arbitrary genus, called fundamental formula and mentioned in §4 of this article, which must include the KdV hierarchy and the KP equations of higher genera but he explicitly presented them only of genus three case. On the other hand, in [BEL1-2], the authors developed a method in terms of matrices by considering a subset of $\wp$-functions $(\wp_{gi})_{\{i=1,\cdots,g\}}$ as a vector and then gave the explicit relation of the KdV hierarchy and the hyperelliptic $\wp$-functions of arbitrary genus g. Their method is consistent with the zero curvature condition in modern soliton theory. Using the hyperelliptic sigma function and defining natural sigma functions of more general algebraic curves, the authors in [BEL1-3,CEEK, EE, EEL, EEP, N] have been constructing deeper theories of abelian functions and soliton equations. Thus it is needless to say that [BEL1-3,CEEK, EE, EEL, EEP, N] are beyond the world of Baker. In fact most of results in §2 of this article (proposition 4 and theorem 6) has been mentioned in their studies [BEL1,2, EE] and review part of Baker's theory in [BEL2] is very nice even for readers who are not familiar with hyperelliptic

functions. In [BEL3], it was pointed out that $\wp_{11}$ of a hyperelliptic curve of genus $g > 2$ with odd degree polynomial is a solution of the KP equation, which corresponds to the relation (IV-15) in (2-15) of this article. However in [BEL1-3,CEEK, EE, EEL, EEP, N], they did not comment upon the paper [B3], which contains interesting and fruitful results from modern point of view as described in §4. Further as far as I know, there has been no study on a hyperelliptic function solution of the KP equation over a hyperelliptic curve with even degree polynomial, which directly reproduces the natural dispersion relations of the KP equation. Connection between modern soliton theory [DKJM] and Baker's theory discussed in §4 is also concerned from viewpoint of the reevaluation. Thus I believe that this article is still important.

§2. Hyperelliptic Solutions of KdV Equations

In this section, we will consider hyperelliptic solutions of the first and second KdV equations in the KdV hierarchy. First we will prepare notations and definitions on this article. Although we mainly deal with a curve with genus three, we give definitions and expressions of hyperelliptic curves with general genus for later convenience. In this article, we will mainly use the conventions of Ônishi [Ô1, Ô2]. We denote the set of complex number by $\mathbb{C}$ and the set of integers by $\mathbb{Z}$.

Notation 1. *We deal with a hyperelliptic curve X_g of genus g ($g > 0$) given by the algebraic equation,*

$$\begin{aligned} y^2 &= f(x) \\ &= \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{2g+1} x^{2g+1} \\ &= (x - c_1)\cdots(x - c_g)(x - c_{g+1})\cdots(x - c_{2g})(x - c_{2g+1}), \end{aligned} \tag{2-1}$$

where $\lambda_{2g+1} \equiv 1$ and λ_j's and c_j's are complex values.

Since we wish to treat the infinite point in this curve, we should embed it in a projective space. However as it is not difficult, we assume that the curve $y^2 = f(x)$ includes the infinite point. Further for simplicity, we also assume that $f(x) = 0$ is not degenerate. We sometimes express a point P in the curve by the affine coordinate (x, y).

Definition 2 [B1 p.195, B2 p.314, B3 p.137, Ô1 p.385-6, Ô2].

(1) *Let us denote the homology of a hyperelliptic curve X_g by*

$$\mathrm{H}_1(X_g, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z}\beta_j, \tag{2-2}$$

where these intersections are given as $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = \delta_{i,j}$.

(2) *The unnormalized differentials of the first kind are defined by,*

$$\omega_1 := \frac{dx}{2y}, \quad \omega_2 := \frac{xdx}{2y}, \quad \cdots, \quad \omega_g := \frac{x^{g-1}dx}{2y}. \tag{2-3}$$

(3) *The unnormalized differentials of the second kind are defined by,*

$$\eta_j := \frac{1}{2y}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}x^k dx, \quad (j=1,\cdots,g). \tag{2-4}$$

(4) *The unnormalized period matrices are defined by,*

$$\mathbf{\Omega}' := \left[\int_{\alpha_j}\omega_i\right], \quad \mathbf{\Omega}'' := \left[\int_{\beta_j}\omega_i\right], \quad \mathbf{\Omega} := \begin{bmatrix}\mathbf{\Omega}' \\ \mathbf{\Omega}''\end{bmatrix}. \tag{2-5}$$

(5) *The normalized period matrices are given by,*

$${}^t[\,\widehat{\omega}_1 \cdots \quad \widehat{\omega}_g\,] := {\mathbf{\Omega}'}^{-1}\ {}^t[\,\omega_1 \quad \cdots\omega_g\,], \quad \mathbb{T} := {\mathbf{\Omega}'}^{-1}\mathbf{\Omega}'', \quad \hat{\mathbf{\Omega}} := \begin{bmatrix}1_g \\ \mathbb{T}\end{bmatrix}. \tag{2-6}$$

(6) *The complete hyperelliptic integrals of the second kind are given as*

$$H' := \left[\int_{\alpha_j}\eta_i\right], \quad H'' := \left[\int_{\beta_j}\eta_i\right]. \tag{2-7}$$

(7) *By defining the Abel map for g-th symmetric product of the curve X_g and for points $\{Q_i\}_{i=1,\cdots,g}$ in the curve,*

$$\begin{aligned}\hat{w} &: \mathrm{Sym}^g(X_g) \longrightarrow \mathbb{C}^g, \quad \left(\hat{w}_k(Q_i) := \sum_{i=1}^g \int_\infty^{Q_i}\hat{\omega}_k\right), \\ w &: \mathrm{Sym}^g(X_g) \longrightarrow \mathbb{C}^g, \quad \left(w_k(Q_i) := \sum_{i=1}^g \int_\infty^{Q_i}\omega_k\right),\end{aligned} \tag{2-8}$$

the Jacobi varieties $\hat{\mathcal{J}}_g$ and $\mathcal{J}_g$ are defined as complex torus,

$$\hat{\mathcal{J}}_g := \mathbb{C}^g/\hat{\mathbf{\Lambda}}, \quad \mathcal{J}_g := \mathbb{C}^g/\mathbf{\Lambda}. \tag{2-9}$$

Here $\hat{\mathbf{\Lambda}}$ ($\mathbf{\Lambda}$) is a lattice generated by $\hat{\mathbf{\Omega}}$ ($\mathbf{\Omega}$).

(8) *We defined the theta function over $\mathbb{C}^g$ characterized by $\hat{\mathbf{\Lambda}}$,*

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z) := \theta\begin{bmatrix}a\\b\end{bmatrix}(z;\mathbb{T}) := \sum_{n\in\mathbb{Z}^g}\exp\left[2\pi i\left\{\frac{1}{2}\ {}^t(n+a)\mathbb{T}(n+a) + \ {}^t(n+a)(z+b)\right\}\right], \tag{2-10}$$

for g-dimensional vectors a and b.

We should note that these contours in the integrals are, for example, given in p.3.83 in [M]. Thus above values can be, in principle, computed in terms of numerical method for a given $y^2 = f(x)$.

It is also noted that on (2-3), we have employed the convention of Ônishi [Ô1, Ô2], which differs from Baker's original one by factor 1/2. Due to the difference, the results and definitions in [B1, B2, B3] will be slightly modified but the factor set us free from extra constant factors in various situations [G, Ô1, Ô2, Ô3].

Definition 3 ($\wp$-function, Baker) [B1, B2 p.336, p.358, p.370, Ô1 p.386-7, Ô2].

We prepare the coordinate in $\mathbb{C}^g$ for points $(x_i, y_i)_{i=1,\cdots,g}$ of the curve $y^2 = f(x)$,

$$u_j := \sum_{i=1}^{g} \int_{\infty}^{(x_i,y_i)} \omega_j. \tag{2-11}$$

(1) *Using the coordinate u_j, sigma function, which is a holomorphic function over $\mathbb{C}^g$, is defined by*

$$\sigma(u) = \sigma(u; X_g) := \exp(-\frac{1}{2}\ {}^t\ uH'\mathbf{\Omega}'^{-1}u)\vartheta\begin{bmatrix}\delta'' \\ \delta'\end{bmatrix}(\mathbf{\Omega}'^{-1}u; \mathbb{T}). \tag{2-12}$$

where

$$\delta' = {}^t\begin{bmatrix}\frac{g}{2} & \frac{g-1}{2} & \cdots & \frac{1}{2}\end{bmatrix}, \quad \delta'' = {}^t\begin{bmatrix}\frac{1}{2} & \cdots & \frac{1}{2}\end{bmatrix}. \tag{2-13}$$

(2) *In terms of σ function, $\wp$-function over the hyperelliptic curve is given by*

$$\wp_{\mu\nu}(u) = -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \sigma(u). \tag{2-14}$$

The σ-function is a well-tuned theta-function. (2-13) is related to so-called Riemannian constant K as mentioned in p.3.80-82 in [M]; $\delta' + \mathbb{T}\delta''$ agrees with K. As the σ-function [B2, p.336, p.358] consists of the shifting Riemann theta function (2-10) [B2, p.324, p.336], the Riemann constant K outwardly disappears. (Thus the σ-function vanishes just over the theta divisor.) Using the σ-function, Baker derived the multiple relations of $\wp$-functions and so on. Hereafter we assume that genus of the curve is three.

Proposition 4 [B3 p.155-6,Ô1 p.388,Ô2].

Let us express $\wp_{\mu\nu\rho} := \partial\wp_{\mu\nu}(u)/\partial u_\rho$ and $\wp_{\mu\nu\rho\lambda} := \partial^2\wp_{\mu\nu}(u)/\partial u_\mu \partial u_\nu$. Then hyperelliptic $\wp$-functions obey the relations,

$(IV-1) \quad \wp_{3333} - 6\wp_{33}^2 = 2\lambda_5\lambda_7 + 4\lambda_6\wp_{33} + 4\lambda_7\wp_{32},$

$$
\begin{aligned}
&(IV-2) && \wp_{3332} - 6\wp_{33}\wp_{32} = 4\lambda_6\wp_{32} + 2\lambda_7(3\wp_{31} - \wp_{22}),\\
&(IV-3) && \wp_{3331} - 6\wp_{31}\wp_{33} = 4\lambda_6\wp_{31} - 2\lambda_7\wp_{21},\\
&(IV-4) && \wp_{3322} - 4\wp_{32}^2 - 2\wp_{33}\wp_{22} = 2\lambda_5\wp_{32} + 4\lambda_6\wp_{31} - 2\lambda_7\wp_{21},\\
&(IV-5) && \wp_{3321} - 2\wp_{33}\wp_{21} - 4\wp_{32}\wp_{31} = 2\lambda_5\wp_{31},\\
&(IV-6) && \wp_{3311} - 4\wp_{31}^2 - 2\wp_{33}\wp_{11} = 2\Delta,\\
&(IV-7) && \wp_{3222} - 6\wp_{32}\wp_{22} = -4\lambda_2\lambda_7 - 2\lambda_3\wp_{33} + 4\lambda_4\wp_{32} + 4\lambda_5\wp_{31} - 6\lambda_7\wp_{11},\\
&(IV-8) && \wp_{3221} - 4\wp_{32}\wp_{21} - 2\wp_{31}\wp_{22} = -2\lambda_1\lambda_7 + 4\lambda_4\wp_{31} - 2\Delta,\\
&(IV-9) && \wp_{3211} - 4\wp_{31}\wp_{21} - 2\wp_{32}\wp_{11} = -4\lambda_0\lambda_7 + 2\lambda_3\wp_{31},\\
&(IV-10) && \wp_{3111} - 6\wp_{31}\wp_{11} = 4\lambda_0\wp_{33} - 2\lambda_1\wp_{32} + 4\lambda_2\wp_{31},\\
&(IV-11) && \wp_{2222} - 6\wp_{22}^2 = -8\lambda_2\lambda_6 + 2\lambda_3\lambda_5\\
& && - 6\lambda_1\lambda_7 - 12\lambda_2\wp_{33} + 4\lambda_3\wp_{32} + 4\lambda_4\wp_{22} + 4\lambda_5\wp_{21} - 12\lambda_6\wp_{11} + 12\Delta,\\
&(IV-12) && \wp_{2221} - 6\wp_{22}\wp_{21} = -4\lambda_1\lambda_6 - 8\lambda_0\lambda_7 - 6\lambda_1\wp_{33} + 4\lambda_3\wp_{31} + 4\lambda_4\wp_{21} - 2\lambda_5\wp_{11},\\
&(IV-13) && \wp_{2211} - 4\wp_{21}^2 - 2\wp_{22}\wp_{11} = -8\lambda_0\lambda_6 - 8\lambda_0\wp_{33} - 2\lambda_1\wp_{32} + 4\lambda_2\wp_{31} + 2\lambda_3\wp_{21},\\
&(IV-14) && \wp_{2111} - 6\wp_{21}\wp_{11} = -2\lambda_0\lambda_5 - 8\lambda_0\wp_{32} + 2\lambda_1(3\wp_{31} - \wp_{22}) + 4\lambda_2\wp_{21},\\
&(IV-15) && \wp_{1111} - 6\wp_{11}^2 = -4\lambda_0\lambda_4 + 2\lambda_1\lambda_3 + 4\lambda_0(4\wp_{31} - 3\wp_{22}) + 4\lambda_1\wp_{21} + 4\lambda_2\wp_{11},
\end{aligned}
\tag{2-15}
$$

where

$$\Delta = \wp_{32}\wp_{21} - \wp_{31}\wp_{22} + \wp_{31}^2 - \wp_{33}\wp_{11}. \tag{2-16}$$

Remark 5.

(1) *Due to the definitions, indices of* $\wp$ *are symmetric, i.e.,* $\wp_{\mu\nu} = \wp_{\nu\mu}$, $\wp_{\mu\nu\rho} = \wp_{\rho\mu\nu} = \wp_{\nu\rho\mu}$ *and so on.*

(2) *Above equations are independent because the axes of Jacobian* $\mathcal{J}_g$ *are independent.*

(3) *As Baker did [B3, p.151], introducing the bilinear differential operator* $\mathbf{D}_\nu$,

$$\mathbf{D}_\mu\sigma(u)\sigma(u) := (\frac{\partial}{\partial u'_\mu} - \frac{\partial}{\partial u_\mu})\sigma(u')\sigma(u)|_{u=u'}, \tag{2-17}$$

we have the relations,

$$\wp_{\mu\nu} = -\frac{1}{2\sigma^2}\mathbf{D}_\mu\mathbf{D}_\nu\sigma\sigma, \tag{2-18}$$

$$\wp_{\lambda\mu\nu\rho} - 2(\wp_{\mu\nu}\wp_{\lambda\rho} + \wp_{\nu\lambda}\wp_{\rho\mu} + \wp_{\lambda\mu}\wp_{\rho\nu}) = -\frac{1}{2\sigma^2}\mathbf{D}_\lambda\mathbf{D}_\mu\mathbf{D}_\nu\mathbf{D}_\rho\sigma\sigma. \tag{2-19}$$

Then the equations in Proposition 4 can be regarded as the bilinear equations of σ*-functions. For example, (IV-1) is given by*

$$(\mathbf{D}_3^4 - 4\lambda_6\mathbf{D}_3^2 - 4\mathbf{D}_3\mathbf{D}_2 - 4\lambda_5\lambda_7)\sigma\sigma = 0. \tag{2-20}$$

Theorem 6.

For $v = -2(\wp_{33} + \lambda_6/3)$ *and* $v(t_1, t_3, t_5) = v(u_3, -\dfrac{u_2}{2^2}, \dfrac{u_1}{2^4} + \dfrac{3}{2^4\lambda_6}u_2)$ *obeys first and second KdV equations:*

$$\partial_{t_3} v + 6v\partial_{t_1} v + \partial_{t_1}^3 v = 0, \tag{2-21}$$

$$\partial_{t_5} v + 30v^2\partial_{t_1} v + 20\partial_{t_1} v\partial_{t_1}^2 v + 10v\partial_{t_1}^3 v + \partial_{t_1}^5 v = 0. \tag{2-22}$$

Proof. By differentiating (IV-1) in u_3 and tuning them, we obtain the KdV equation. We note that second KdV equation is expressed by

$$\partial_{t_5} v + (\partial_{t_1}^2 + 2v + 2\partial_{t_1} v\partial_{t_1}^{-1})(6v\partial_{t_1} v + \partial_{t_1}^3 v) = 0, \tag{2-23}$$

where $\partial_{t_1}^{-1}$ implies an integral with respect to t_1. By setting $2\partial_{u_3} \times (\mathrm{IV}-2) + \partial_{u_2} \times (\mathrm{IV}-1)$ and $\partial_{t_5} = 16\partial_{u_1} + \dfrac{16\lambda_2}{3}\partial_{u_2}$, we obtain second KdV equation.∎

Remark 7.

(1) *Theorem 6 and definition of* $\wp$ *mean that solutions of the KdV equation are explicitly constructed. The quantities in definitions 2 and 3 can be, in principle, evaluated in terms of numerical computations because there is no ambiguous parameter.*

(2) *We note the dispersion relations:* u_j *behaves like* $(1/\overline{x})^{2(g-j)+1}$ *around* ∞ *point if we use local coordinate* $\overline{x}^2 := x$. *By comparing the order of* $\overline{x}$ *denoted by* $ord_{\overline{x}}$, *we have the relations,*

$$ord_{\overline{x}}(u_2) = 3ord_{\overline{x}}(u_3), \quad ord_{\overline{x}}(u_1) = 5ord_{\overline{x}}(u_3). \tag{2-24}$$

These are the dispersion relations of the KdV equations.

(3) *Roughly speaking integrating the KdV equation in* t_1 *becomes (IV -1) in proposition 4. Then there appears an undetermined integral constant. However in proposition 4, it is fixed and associated with the coefficients of the algebraic equation* $y^2 = f(x)$. *Thus (IV-1) in proposition 4 is more fundamental than the KdV equation.*

(4) *For genus two case: we put that* $\partial\sigma/\partial u_3 = 0$ *and* $\lambda_6 = \lambda_7 = 0$; *(IV-1)-(IV-10) becomes meaningless* $0 = 0$ *and* $\Delta = 0$. $v = -2(\wp_{22} + \lambda_4/3)$ *and* $v(t_1, t_3) = v(u_2, -\dfrac{u_1}{2^2})$ *obeys first KdV equation (2-21).*

(5) *For genus one case or elliptic functions case: we put that* $\partial\sigma/\partial u_\mu = 0$ $(\mu = 2, 3)$, *and* $\lambda_a = 0$ $(a = 4, 5, 6, 7)$; *only (IV-15) survives, which is the relation of elliptic* $\wp$ *function.*

§3. Hyperelliptic Solutions of KP Equation

Instead of the curve of $(2g+1)$-degree, we will deal with a hyperelliptic curve of $(2g+2)$-degree in this section.

Notation 8.

$$\begin{aligned} y^2 &= \overline{f}(x) \\ &= \overline{\lambda}_0 + \overline{\lambda}_1 x + \bar{\lambda}_2 x^2 + \cdots + \bar{\lambda}_{2g+2} x^{2g+2} \\ &= (x-\alpha_1)\cdots(x-\alpha_g)(x-\alpha_{g+1})\cdots(x-\alpha_{2g})(x-\alpha_{2g+1})(x-\alpha_{2g+2}), \end{aligned} \tag{3-1}$$

where $\bar{\lambda}_{2g+2} \equiv 1$ *and* $\bar{\lambda}_j$*'s and* α_j*'s are complex values.*

Remark 9 [B1 p.195, B3 p.144-5].

(1) *The transformation between* $y^2 = f(x)$ *and* $\zeta^2 = \overline{f}(\xi)$ *is as follows*

$$x = \frac{a}{\xi - \alpha_{2g+2}}, \quad c_i = \frac{a}{\alpha_i - \alpha_{2g+2}}, \quad \zeta = \frac{(\xi - \alpha_{2g+2})^{g+1}}{-4\prod_i^{2g-1} c_j} y. \tag{3-2}$$

(2) *The unnormalized differentials of the first kind are defined by,*

$$\omega_1 = \frac{dx}{2y}, \quad \omega_2 = \frac{xdx}{2y}, \quad \cdots, \quad \omega_g = \frac{x^{g-1}dx}{2y}. \tag{3-3}$$

(3) *The unnormalized differentials of the second kind are defined by ([B2, p.195]),*

$$\eta_j = \frac{1}{2y}\sum_{k=j}^{2g+1-j}(k+1-j)\bar{\lambda}_{k+1+j}x^k dx, \quad (j = 1, \cdots, g). \tag{3-4}$$

(4) *The definition of* σ *and* $\wp$*-functions are the same as those in definition 2 and 3, where we regard that* y *obeys the equation* $y^2 = \overline{f}(x)$ *instead of* $y^2 = f(x)$*.*

Proposition 10 [B3 p.155-6].

The hyperelliptic $\wp$*-functions of a curve* $y^2 = \overline{f}(x)$ $(g = 3)$ *obey the relations*

$$\begin{aligned}
&(X-1) \quad \wp_{3333} - 6\wp_{33}^2 = 2\overline{\lambda}_5\overline{\lambda}_7 + 4\overline{\lambda}_6\wp_{33} + 4\overline{\lambda}_7\wp_{32} - 8\overline{\lambda}_4\overline{\lambda}_8 + 4\overline{\lambda}_8(4\wp_{31} - 3\wp_{22}),\\
&(X-2) \quad \wp_{3332} - 6\wp_{33}\wp_{32} = 4\overline{\lambda}_6\wp_{32} + 2\overline{\lambda}_7(3\wp_{31} - \wp_{22}) - 4\overline{\lambda}_3\overline{\lambda}_8 + 8\overline{\lambda}_8\wp_{21},\\
&(X-3) \quad \wp_{3331} - 6\wp_{31}\wp_{33} = 4\overline{\lambda}_6\wp_{31} - 2\overline{\lambda}_7\wp_{21} + 4\overline{\lambda}_8\wp_{11},\\
&(X-4) \quad \wp_{3322} - 4\wp_{32}^2 - 2\wp_{33}\wp_{22} = 2\overline{\lambda}_5\wp_{32} + 4\overline{\lambda}_6\wp_{31} - 2\overline{\lambda}_7\wp_{21} - 8\overline{\lambda}_2\overline{\lambda}_8 - 8\overline{\lambda}_8\wp_{11},\\
&(X-5) \quad \wp_{3321} - 2\wp_{33}\wp_{21} - 4\wp_{32}\wp_{31} = 2\overline{\lambda}_5\wp_{31} - 4\overline{\lambda}_1\overline{\lambda}_8,\\
&(X-6) \quad \wp_{3311} - 4\wp_{31}^2 - 2\wp_{33}\wp_{11} = 2\Delta,\\
&(X-7) \quad \wp_{3222} - 6\wp_{32}\wp_{22} = -4\overline{\lambda}_2\overline{\lambda}_7 - 2\overline{\lambda}_3\wp_{33} + 4\overline{\lambda}_4\wp_{32} + 4\overline{\lambda}_5\wp_{31} - 6\overline{\lambda}_7\wp_{11} - 8\overline{\lambda}_1\overline{\lambda}_8,\\
&(X-8) \quad \wp_{3221} - 4\wp_{32}\wp_{21} - 2\wp_{31}\wp_{22} = -2\overline{\lambda}_1\overline{\lambda}_7 + 4\overline{\lambda}_4\wp_{31} - 2\Delta - 8\overline{\lambda}_0\overline{\lambda}_8,
\end{aligned} \tag{3-5}$$

(9)-(15) and Δ *which have the same form as those in proposition 4 by replacing* λ*'s with* $\overline{\lambda}$*'s.*

Theorem 11.

For $v = -2(\wp_{33} + \overline{\lambda}_6/3)$ *and* $u(t_1, t_2, t_3) = v(u_3, \dfrac{u_2}{2\sqrt{-3}}, -\dfrac{u_1}{2^4} - \dfrac{3}{2^2\overline{\lambda}_7}u_2)$ *obeys the KP equation:*

$$\partial_{t_1}(\partial_{t_3}v + 6v\partial_{t_1}v + \partial_{t_1}^3 v) = \partial_{t_2}^2 v. \tag{3-6}$$

Proof. Noting $\overline{\lambda}_8 = 1$, direct substitution of them into (3-6) is differential of (X-1) in u_3. ∎

Remark 12.

(1) *Theorem 11 means that we obtain an explicit function form of hyperelliptic function solution of the KP equation.*

(2) *We note the dispersion relation. Since the curve* $y^2 = \overline{f}(x)$ *is not ramified at infinity point. There* u_j *behaves like* $(1/x)^{(g-j)}$ *upto a constant factor. By comparing the order of* x *denoted by* ord_x*, we have the relations,*

$$ord_x(u_2) = 2ord_x(u_3), \quad ord_x(u_1) = 3ord_x(u_3). \tag{3-7}$$

These are the dispersion relations of the KP equation.

§4. Discussion

Since derivation of proposition 10 is essentially the same as that of proposition 4, we will give a sketch only of the derivation of the differential equations in proposition 4 and comment upon its relation to the soliton theory.

Definition 13 [B1 p.195, B2 p.314, p.335-6, Ô2].

For points of $\mathrm{P}(x, y)$, $\mathrm{Q}(z, w)$, $\mathrm{A}(a, b)$, $\mathrm{B}(c, d)$ *over* X_g*, we introduce the quantities,*

(1)

$$\mathbf{R}_{\mathrm{Q,B}}^{\mathrm{P,A}} := \int_{\mathrm{A}}^{\mathrm{P}}\int_{\mathrm{B}}^{\mathrm{Q}} \frac{f(x, z) + 2yw}{(x - z)^2}\frac{dx}{2y}\frac{dz}{2w}, \tag{4-1}$$

where

$$f(x, z) := \sum_{j=0}^{g} x^j z^j (\lambda_{2j+1}(x + z) + 2\lambda_{2j}). \tag{4-2}$$

(2)

$$\mathbf{P}_{\mathrm{Q,B}}^{\mathrm{P,A}} := \int_{\mathrm{A}}^{\mathrm{P}} \left(\frac{y + w}{x - z} - \frac{y + d}{x - c}\right)\frac{dx}{2y}. \tag{4-3}$$

Proposition 14 [B1 p.194-5, B2 p.318, p.336, Ô2].

(1) $\mathbf{R}_{\mathrm{Q,B}}^{\mathrm{P,A}}$ *and* $\mathbf{P}_{\mathrm{Q,B}}^{\mathrm{P,A}}$ *as a function of* P *have singularity around* $\mathrm{P} = \mathrm{Q}$, B *of first order with the residues* $1, -1$ *and holomorphic otherwise. In other words, they are unnormalized third differentials.*

(2)

$$\mathbf{R}^{\mathrm{P},\mathrm{A}}_{\mathrm{Q},\mathrm{B}} = \int_{\mathrm{A}}^{\mathrm{P}} \omega_1 \int_{\mathrm{B}}^{\mathrm{Q}} \eta_1 + \cdots + \int_{\mathrm{A}}^{\mathrm{P}} \omega_g \int_{\mathrm{B}}^{\mathrm{Q}} \eta_g + \mathbf{P}^{\mathrm{P},\mathrm{A}}_{\mathrm{Q},\mathrm{B}}. \tag{4-4}$$

(3) *For* P_j, $\mathrm{Q}_j \in X$, *(*$j = 1, \cdots, g$*), and*

$$u = \sum_{j=1}^{g} \int_{\infty}^{\mathrm{P}_j} \omega, \quad u' = \sum_{j=1}^{g} \int_{\infty}^{\mathrm{Q}_j} \omega, \tag{4-5}$$

the following relation holds,

$$\exp\left(\sum_{j=1}^{g} \mathbf{R}^{\mathrm{P},\mathrm{Q}}_{\overline{\mathrm{P}_j},\overline{\mathrm{Q}_j}}\right) = \frac{\sigma\left(\int_{\infty}^{\mathrm{P}} \omega + u\right)\sigma\left(\int_{\infty}^{\mathrm{Q}} \omega + u'\right)}{\sigma\left(\int_{\infty}^{\mathrm{P}} \omega + u'\right)\sigma\left(\int_{\infty}^{\mathrm{Q}} \omega + u\right)}, \tag{4-6}$$

where $\overline{\mathrm{P}_j}$ $(\overline{\mathrm{Q}_j})$ *is conjugate of* P_j (Q_j) *with respect to the symmetry of hyperelliptic curve* $(x, y) \to (x, -y)$.

Remark 15.

The relation 4-6 is very important. It holds for appropriate σ*-functions and third differentials in a general compact Riemannian surface [B1 p.290], even though their form can not globally written like definition 13. As we show below, the relation plays important roles in both Baker's theory and DKJM-theory [DKJM].*

Here we will sketch the derivation of the equations in propositions 4 following [B1] and [B3]. First we introduce the variables for the divisors $\mathrm{P}_j = (x_j, y_j)$ and $\mathrm{P} = (x, y) \equiv (x_0, y_0)$ in notations in proposition 14 (3),

$$\mathsf{t} := (\int_{\infty}^{P} \omega + u), \tag{4-7}$$

$$R(z) := (z - x_0)F(z) := (z - x_0)(z - x_1)(z - x_2)\cdots(z - x_g), \tag{4-8}$$

$$\frac{R(z)}{(z - x_r)(z - x_s)} =: z^{g-1} + c_1^{r,s} z^{g-2} + c_2^{r,s} z^{g-3} + \cdots + c_g^{r,s}, \tag{4-9}$$

and for generic parameter e,

$$\overline{\delta}_e := \sum_{\mu=1}^{g} e^{\mu-1} \frac{\partial}{\partial \mathsf{t}_\mu}. \tag{4-10}$$

We operate $\overline{\delta}_{e_1}\overline{\delta}_{e_2}$ to the both sides in the relation (4-6) in proposition 14. We should note the relation,

$$\sum_{r=0, r\neq s}^{g} \frac{x_r - x_s}{R'(x_r)} c_{l-1}^{r,s} x_r^{g-k} = \delta_l^k, \tag{4-11}$$

where $c_0^{r,s} = 1$ and $R'(x_r) = dR(z)/dz|_{z=x_r}$. By taking limit $x_0 \to \infty$, we obtain [B1 p.328, p.376]

$$(e_1 - e_2)^2 \sum_{\lambda=1}^{g} \sum_{\mu=1}^{g} \wp_{\lambda\mu}(u) e_1^{\lambda-1} e_2^{\mu-1}$$
$$= \left(\sum_{r=1,s=1}^{g} \frac{F(e_1)F(e_2)(2y_r y_s - f(x_r, x_s))}{(e_1 - x_r)(e_2 - x_r)(e_1 - x_s)(e_2 - x_s)F'(x_r)F'(x_s)} \right). \tag{4-12}$$

We deform it by shifting the zero of $\wp$ to obtain [B1, p.328, B3 p.138],

$$\sum_{\lambda=1}^{g} \sum_{\mu=1}^{g} \wp_{\lambda\mu}(u) e_1^{\lambda-1} e_2^{\mu-1} = F(e_1)F(e_2) \left(\sum_{r=1}^{g} \frac{y_r}{(e_1 - x_r)(e_2 - x_r)F'(x_r)} \right)^2$$
$$- \frac{f(e_1)F(e_2)}{(e_1 - e_2)^2 F'(e_1)} - \frac{f(e_2)F(e_1)}{(e_1 - e_2)^2 F'(e_2)} + \frac{f(e_1, e_2)}{(e_1 - e_2)^2}. \tag{4-13}$$

Even though in [B3] Baker adopted this formula (4-13) as a definition of $\wp$-functions, his arguments on this formula stood upon the background of so many studies on the hyperelliptic function [B1, B2]. Thus we should regard (4-13) as a theorem which was proved in [B1].

Introducing another operator,

$$\delta_e = \frac{1}{F(e)} \sum_{j=1}^{g} e^{j-1} \frac{\partial}{\partial u_j}, \tag{4-14}$$

we operate $\delta_{e_3}\delta_{e_4}$ to above relation (4-13) and then we have "fundamental formula" [B3, p.144]. The section I in [B3] devoted the derivation of his fundamental formula, which is very tedious and complex but somewhat attractive. In fact, tracing his derivations makes me feel that there might be deep symmetry behind his theory. In section II in [B3], Baker concentrated genus three case. By comparing the coefficients of each $e_1^a e_2^b e_3^c e_4^d$, he discovered the differential equations in propositions 4 and 10. In the comparison, Baker used the symmetric functions, Pfaffian and bilinear operators. The symmetric functions naturally appears because the differential of the first kind in the hyperelliptic curve is expressed by [B3],

$$\begin{pmatrix} du_1 \\ du_2 \\ du_3 \\ . \\ . \\ du_g \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1/y_1 & 1/y_2 & \cdots & 1/y_g \\ x_1/y_1 & x_2/y_2 & \cdots & x_g/y_g \\ x_1^2/y_1 & x_2^2/y_2 & \cdots & x_g^2/y_g \\ . & . & \cdots & . \\ . & . & \cdots & . \\ x_1^{g-1}/y_1 & x_2^{g-1}/y_2 & \cdots & x_g^{g-1}/y_g \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ . \\ . \\ dx_g \end{pmatrix}. \tag{4-15}$$

This matrix is resemble to Vandermonde matrix. In fact (4-11) is an identity used in construction of inverse matrix of Vandermonde matrix.

Corresponding to the above matrix (4-15), behavior of differentials of the second kind in theory of KP hierarchy [K1, K2, SS, DKJM, KNTY] is sometimes determined by the Vandermonde matrix,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \overline{x}_1 & \overline{x}_2 & \cdots & \overline{x}_p \\ \overline{x}_1^2 & \overline{x}_2^2 & \cdots & \overline{x}_p^2 \\ . & . & \cdots & . \\ \overline{x}_1^{p-1} & \overline{x}_2^{p-1} & \cdots & \overline{x}_p^{p-1} \end{pmatrix}. \tag{4-16}$$

The difference between Baker's theory of hyperelliptic function and modern soliton theory could be regarded as the difference between (4-15) and (4-16).

In modern soliton theory [SS, DKJM, KNTY], we deal with a formal graded ring $G\mathbb{C}[[\overline{x}]] := \cup_n G^n\mathbb{C}[[\overline{x}]]$ related to degrees of $\overline{x}$'s as a localized ring at infinity point of an algebraic curve. Then we consider maps among quotient modules $G^n\mathbb{C}[[\overline{x}]]\ /G^{n-1}\mathbb{C}[[\overline{x}]]$, which consists of $\partial_{\overline{x}}$ and $\overline{x}$. The differential ring generated by $\partial_{\overline{x}}$ and $\overline{x}$ becomes Sato theory [SS] and conformal field theory [KNTY] after appropriately modifying it. There naturally appear the Vandermonde matrix (4-16) of $\overline{x}$'s, symmetric functions, Pfaffian related to behavior of differential of the second kind around the infinity point; the Vandermonde determinate is related to Fermion amplitude [DKJM, KNTY].

In the theory of differentials of the second kind, when one determines the global behavior of algebraic function on a curve by its local data around infinity point, he uses the properties of holomorphic functions over the curves, such as existence theorem, flabby of related sheaves and so on. On the other hand, Baker's theory is of differentials of the first kind and it is a global theory because differentials of the first kind are holomorphic allover the curve and explicitly given. Accordingly we can deal with the hyperelliptic functions in the framework of Baker's theory as we do with elliptic functions.

We will comment on the proposition 4 in the framework of DKJM-theory [DKJM].

Remark 16.

For points $P = (x, y)$, $Q = (\sqrt{-1}x, y)$ and $\overline{P} = (x, -y)$ around the infinity points $x = \overline{x}^2$, we obtain the following relations

(1)

$$\mathbf{R}^{\mathrm{P,Q}}_{\mathrm{A},\overline{\mathrm{B}}} = \mathbf{R}^{\overline{\mathrm{A}},\overline{\mathrm{B}}}_{\mathrm{P,Q}} \tag{4-17}$$

(2)

$$\int_\infty^{(x,y)} \omega_\mu = -\int_\infty^{(\sqrt{-1}x,y)} \omega_\mu = \int_{(\sqrt{-1}x,y)}^{(x,y)} \frac{x^{\mu-1}dx}{y}$$
$$= -\frac{1}{2g-2\mu+1}\frac{1}{\overline{x}^{2g-2\mu+1}} + \text{ lower oder terms.} \tag{4-18}$$

(3)

$$\int_{(\sqrt{-1}x,y)}^{(x,y)} \eta_j = 2[\overline{x}^{2g-2j+1}] + \text{ lower oder terms.} \tag{4-19}$$

(4)

$$\sum_{j=1}^{g} \mathbf{R}^{\mathrm{P},\mathrm{Q}}_{\overline{\mathrm{P}_j,\mathrm{Q}_j}} = -2[(u_1-u_1')\overline{x}^{2g-1}+(u_2-u_2')\overline{x}^{2g-3}+\cdots+(u_g-u_g')\overline{x}] + \sum_{j=1}^{g} \mathbf{P}^{\mathrm{P},\mathrm{Q}}_{\overline{\mathrm{P}_j,\mathrm{Q}_j}} \\ + \text{ lower oder terms.} \tag{4-20}$$

Using the remark 16 and setting $g=\infty$, the relation (3) in proposition 14 is reduced to the generating relation of the KdV hierarchy in DKJM-method:

$$\oint_\infty \frac{d\overline{x}}{\overline{x}} \exp(\sum_{j=1}^{g}(u_j-u_j')\overline{x}^{2g-2i+1})\sigma(u_1-\frac{1}{2g-1}\frac{1}{\overline{x}^{2g-1}}, u_2-\frac{1}{2g-3}\frac{1}{\overline{x}^{2g-3}},\cdots,u_g-\frac{1}{\overline{x}}) \\ \sigma(u_1'+\frac{1}{2g-1}\frac{1}{\overline{x}^{2g-1}}, u_2'+\frac{1}{2g-3}\frac{1}{\overline{x}^{2g-3}},\cdots,u_g'+\frac{1}{\overline{x}})=0. \tag{4-21}$$

In terms of differential operators, we can rewrite this relation and then we obtain the KdV hierarchy [DKJM]. Thus the origins of the KdV hierarchy in Baker's method and DKJM-method are the same.

Remark 17.

We will summary the difference between the soliton theory and Baker's theory.

(1) *As in soliton-theory of the KdV hierarchy [DKJM, K1, K2, SS], we investigate the behavior of meromorophic functions around infinity point of a hyperelliptic curve, 1-1) it can be regarded as a theory of differentials of the second kind, 1-2) it can be extended to theory of meromorophic functions of a general compact Riemannian surface as the theory of the KP hierarchy [DKJM, K1, K2, KNTY, SS], and 1-3) we can not determine fine structure of meromorophic functions of non-degenerate curve.*

(2) *As in the Baker's theory of hyperelliptic $\wp$-functions, we consider behavior of $\wp$-functions around generic points (x_1, y_1), $\cdots$ (x_g, y_g) of a hyperelliptic curve, 2-1) we directly deal with differentials of first kind which are holomorphic allover the curve, 2-2) we can determine all parameters in $\wp$-functions of the curve, 2-3) we can give explicit function forms of $\wp$-functions and coefficients of Laurent expansions around any points in the curve, and 2-4) we can not extend it to general compact Riemannian surface with this concreteness.*

(3) *The differentials of the first kind and the second kinds are complementarily connected as the term in (4-4) of the most important identity (4-6). Thus in (4-6), they behaves like two sides of the same coin.*

Finally we comment upon this study. In Baker's theory, we have no ambiguous and dependent parameters while in ordinary soliton theory of periodic solutions there appear undetermined parameters which must satisfy several relations. Hirota and Ito gave explicit function forms of hyperelliptic functions of genera two and three as periodic solutions of the KdV equation (2-21) [HI]; they determined several parameters by means of numerical computations. However functions should be expressed only by independent variables and thus Baker's theory has the advantage and is appropriate even from viewpoint of numerical study. I hope that in near future, anyone would be able to plot graph of any hyperelliptic functions or any periodic multi-soliton solutions like graphs in [HI], using a personal computer and a laser printer, as we can do for elliptic functions or elliptic soliton solutions.

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